CP^{N-1} harmonic maps and the Weierstrass problem

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A Weierstrass-type system of equations corresponding to the CP^{N-1} harmonic maps is presented. The system constitutes a further generalization of our previous construction [J. Math. Phys. **44**, 328 (2003)]. It consists of four first order equations for three complex functions which are shown to be equivalent to the CP^{N-1} harmonic maps. When the harmonic maps are holomorphic (or antiholomorphic) one of the functions vanishes and the system reduces to the previously given generalization of the Weierstrass problem. We also discuss a possible interpretation of our results and show that in our new case the induced metric is proportional to the total energy density of the map and not only to its holomorphic part, as was the case in the previous generalizations.
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I. INTRODUCTION

A few years ago Konopelchenko, together with his collaborators,[1,2] introduced the subject of Weierstrass representations of surfaces immersed in multidimensional spaces. This has generated quite a lot of interest[3,4] and has led to the connection with the CP^{N-1} harmonic maps. Exploiting this connection, we have recently proposed a generalization of these ideas to the CP^2 (Ref. 5) case and, more recently, managed to generalize it further—to the CP^{N-1} case.[6]

These generalizations lead to the study of immersed surfaces, whose metric is then related to the properties of the corresponding harmonic maps. In the CP^1 case all harmonic maps (from S^2) are holomorphic (or antiholomorphic) and, as the induced metric is characterized by the holomorphic component of the energy, this characterization is complete. This is also the case for the holomorphic CP^{N-1} maps.

In the CP^{N-1} case (for $N>2$) there are harmonic maps which are not holomorphic[7] and for them the above mentioned construction[6] is not complete, as in the general case we would expect the maps to be characterized by the total energy. Hence a further generalization is called for and such a generalization is provided in this article.

In the next section we briefly review the CP^{N-1} harmonic maps (using the formalism as given in Ref. 7) and in the following sections relate these maps to the various versions of the Weierstrass problem.

II. CP^{N-1} HARMONIC MAPS

A. Formulation

The CP^{N-1} models are, in fact, a generalization of the, perhaps the simplest, sigma model, namely, the S^2 model—also called the vector $O(3)$ model. The CP^{N-1} models involve maps from R^2, or S^2 if a nontrivial topology is required, to CP^{N-1}, i.e.,

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$$C\supset\Omega\ni\zeta=\zeta_1+i\zeta_2\mapsto z=(z^1,\ldots,z^N)\in C^N, \tag{1}$$

where the homogeneous coordinates $z=(z^1,\ldots,z^N)$ have the following property:

$$z\sim z'=\lambda z \quad \text{for} \quad \lambda\neq 0.$$

Exploiting this projective invariance we can require that

$$z^\dagger\cdot z=1 \tag{2}$$

holds, where † denotes Hermitian conjugation, and we are still left with the gauge symmetry

$$z\rightarrow z'=ze^{i\phi}, \tag{3}$$

where ϕ is a real-valued function.

It is easiest to define the CP^{N-1} models in terms of the Lagrangian density[7]

$$L=\tfrac{1}{4}(D_\mu z)^\dagger\cdot D_\mu z, \quad z^\dagger\cdot z=1, \tag{4}$$

where the covariant derivatives D_μ act on $z:S^2\rightarrow CP^{N-1}$ according to the formula

$$D_\mu z=\partial_\mu z-(z^\dagger\cdot\partial_\mu z)z. \tag{5}$$

Here the index $\mu=1,2$ denotes ζ_1 and ζ_2. Note that the covariant derivatives $D_\mu z$ transform under the gauge transformation (3)

$$D_\mu z\rightarrow D_\mu z'=(D_\mu z)e^{i\phi}, \tag{6}$$

so that the dependence on the phase ϕ drops out of the Lagrangian density (4) and so the model is really based on CP^{N-1}.

The total Lagrangian is given by

$$\mathcal{L}=\int L\, d\zeta\, d\bar{\zeta} \tag{7}$$

and, if the CP^{N-1} model is defined over S^2, we require that $\mathcal{L}$ is finite.

For the CP^{N-1} sigma model it is convenient to define

$$z=\frac{f}{|f|}, \tag{8}$$

where $|f|=(f^\dagger\cdot f)^{1/2}$. In terms of f the Lagrangian (7) becomes

$$\mathcal{L}=\int \frac{|\bar{\partial}f|^2+|\partial f|^2}{|f|^4}\, d\zeta\, d\bar{\zeta}, \tag{9}$$

where $|\partial f|^2=(\partial f)^\dagger\cdot(\partial f)$ and $|\bar{\partial}f|^2=(\bar{\partial}f)^\dagger\cdot(\bar{\partial}f)$. The Euler–Lagrange equations for f take the form

$$\left(1-\frac{f\otimes f^\dagger}{|f|^2}\right)\left[\partial\bar{\partial}f-\partial f\,\frac{(f^\dagger\cdot\bar{\partial}f)}{|f|^2}-\bar{\partial}f\,\frac{(f^\dagger\cdot\partial f)}{|f|^2}\right]=0, \tag{10}$$

where we have introduced the holomorphic and antiholomorphic derivatives

$$\partial = \frac{\partial}{\partial(\zeta_1 + i\zeta_2)} = \frac{\partial}{\partial\zeta}, \quad \bar{\partial} = \frac{\partial}{\partial(\zeta_1 - i\zeta_2)} = \frac{\partial}{\partial\bar{\zeta}}, \tag{11}$$

and bar denotes complex conjugation.

B. Integrability and first conservation laws

As is well known,[8] Eqs. (10) can be written as a compatibility condition for a set of two linear spectral equations for an N-component auxiliary vector Ψ,

$$\begin{aligned} \partial\Psi &= \frac{2}{1+\lambda}[\partial P, P]\Psi, \\ \bar{\partial}\Psi &= \frac{2}{1-\lambda}[\bar{\partial} P, P]\Psi, \end{aligned} \tag{12}$$

where λ is a spectral parameter and the N by N matrix P is the projector given by

$$P = \frac{1}{|f|^2} f \otimes f^\dagger, \quad P^\dagger = P, \quad P^2 = P. \tag{13}$$

The compatibility conditions for (12) are then

$$[\partial\bar{\partial} P, P] = 0, \tag{14}$$

which, as can be easily checked, are equivalent to Eqs. (10). Note that (14) can be written in the form of a conservation law

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0 \tag{15}$$

or, equivalently, using the tracelessness of the matrix K, as

$$\partial K - \bar{\partial} K^\dagger = 0, \tag{16}$$

where the matrices K and $K^\dagger$ are given by

$$K = [\bar{\partial} P, P] = \frac{\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger}{|f|^2} + \frac{f \otimes f^\dagger}{|f|^4}[(\bar{\partial} f^\dagger \cdot f) - (f^\dagger \cdot \bar{\partial} f)], \qquad \mathrm{Tr}\, K = 0, \tag{17}$$

and consequently

$$K^\dagger = -[\partial P, P] = -\frac{\partial f \otimes f^\dagger - f \otimes \partial f^\dagger}{|f|^2} + \frac{f \otimes f^\dagger}{|f|^4}[(\partial f^\dagger \cdot f) - (f^\dagger \cdot \partial f)].$$

Note that due to the invariance of the Lagrangian (4) under the gauge transformation (3) we can, without any loss of generality, set one of the components of the vector field f, say f_1, to 1. Then, in the CP^1 case, all quantities are expressible through one variable

$$w = \frac{f_2}{f_1} = f_2 \tag{18}$$

and the Euler–Lagrange equations (10) take the form

$$\partial\bar{\partial}\bar{w}-\frac{2\bar{w}}{(1+|w|^2)}\partial w\bar{\partial}\bar{w}=0. \tag{19}$$

C. Further conservation laws

Let us note that our matrix K in (17) is given by

$$K=M+L, \tag{20}$$

where

$$M=(1-P)\frac{\bar{\partial}f\otimes f^\dagger}{|f|^2} \tag{21}$$

and

$$L=-\frac{f\otimes\bar{\partial}f^\dagger}{|f|^2}(1-P). \tag{22}$$

Thus

$$M^\dagger=\frac{f\otimes\partial f^\dagger}{|f|^2}(1-P),\quad \text{and}\quad L^\dagger=-(1-P)\frac{\partial f\otimes f^\dagger}{|f|^2}. \tag{23}$$

Next we note that the matrices M and L, separately, satisfy our conservations laws (16). To see this consider

$$\begin{aligned}\partial M-\bar{\partial}M^\dagger=&-\partial P\frac{\bar{\partial}f\otimes f^\dagger}{|f|^2}+(1-P)\frac{\bar{\partial}\partial f\otimes f^\dagger}{|f|^2}+(1-P)\frac{\bar{\partial}f\otimes\partial f^\dagger}{|f|^2}-(1-P)\frac{\bar{\partial}f\otimes f^\dagger}{|f|^4}\partial|f|^2\\&-\frac{\bar{\partial}f\otimes\partial f^\dagger}{|f|^2}(1-P)-\frac{f\otimes\bar{\partial}\partial f^\dagger}{|f|^2}(1-P)+\frac{f\otimes\partial f^\dagger}{|f|^2}\bar{\partial}P+\frac{f\otimes\partial f^\dagger}{|f|^4}(1-P)\bar{\partial}|f|^2.\end{aligned} \tag{24}$$

But

$$\partial P=\frac{\partial f\otimes f^\dagger}{|f|^2}+\frac{f\otimes\partial f^\dagger}{|f|^2}-\frac{P}{|f|^2}\partial|f|^2 \tag{25}$$

and so we see that all the terms in (24) become

$$\begin{aligned}&(1-P)\left[\partial\bar{\partial}f-\partial f\frac{(f^\dagger\cdot\bar{\partial}f)}{|f|^2}-\bar{\partial}f\frac{(f^\dagger\cdot\partial f)}{|f|^2}\right]\\&\otimes\frac{f^\dagger}{|f|^2}-\frac{f}{|f|^2}\otimes\left[\partial\bar{\partial}f^\dagger-\partial f^\dagger\frac{(\bar{\partial}f^\dagger f)}{|f|^2}-\bar{\partial}f^\dagger\frac{(\partial f^\dagger\cdot f)}{|f|^2}\right](1-P).\end{aligned} \tag{26}$$

However, due to (10), this is zero. Hence we have two separate conservation laws, namely,

$$\partial M=\bar{\partial}M^\dagger \tag{27}$$

and

$$\partial L=\bar{\partial}L^\dagger. \tag{28}$$

Next we consider the explicit form of the entries of the matrices M and L. To do this we introduce

$$F_{ij} = f_i\,\partial f_j - f_j \partial f_i, \tag{29}$$

and

$$G_{ij} = f_i\,\bar{\partial} f_j - f_j \bar{\partial} f_i. \tag{30}$$

Then, using expressions (29) and (30), we can write the entries of the matrices M and L, equivalently, in the form

$$M_{ij} = \overline{\Phi_i^2}\,\overline{f_j} \tag{31}$$

and

$$L_{ij} = - f_i \overline{\varphi_j^2}, \tag{32}$$

where we have introduced

$$\varphi_i^2 = \frac{1}{A^2}\overline{f_k} F_{ki}, \quad A = \overline{f_l} f_l \tag{33}$$

and

$$\Phi_i^2 = \frac{1}{A^2} f_k \overline{G_{ki}}, \tag{34}$$

and we have used the convention of implicit summation over repeated indices.

Note that from Eqs. (30), (33), and (34) we have two algebraic constraints, namely,

$$\overline{f_k}\,\varphi_k^2 = 0, \quad f_k \Phi_k^2 = 0, \tag{35}$$

which imply that only $(N-1)$ functions φ_i^2 and $(N-1)$ functions Φ_i^2 are linearly independent. So in our further discussion we take as independent functions $\varphi_2^2, \ldots, \varphi_N^2$ and $\Phi_2^2, \ldots, \Phi_N^2$.

Making use of the symmetry (3) we can set, without any loss of generality, say, $f_1 = 1$, and so we end up with the expressions [for (33), and (34)]

$$\begin{aligned} \varphi_i^2 &= \frac{1}{A^2}[(1+f_k\overline{f_k})\,\partial f_i - f_i(\overline{f_k}\,\partial f_k)], \\ \overline{\Phi_i^2} &= \frac{1}{A^2}[(1+f_k\overline{f_k})\,\bar{\partial} \overline{f_i} - \overline{f_i}(\overline{f_k}\,\bar{\partial} f_k)], \quad i = 2, \ldots, N, \end{aligned} \tag{36}$$

where

$$A = 1 + |f_2|^2 + \cdots + |f_N|^2.$$

Note that now all the sums over repeated indices run over $k = 2, \ldots, N$.

Next we invert expressions (36) and so express all the derivatives ∂f_i in terms of φ_i^2's and f_i. This way we find that

$$\partial f_i = A\,[\varphi_i^2 + f_i\,\overline{f_k}\varphi_k^2]. \tag{37}$$

Thus, in particular, for the CP^1 case, Eqs. (37) become

$$\partial f_2 = A^2 \varphi_2^2, \quad A = 1+|f_2|^2,$$

and f_2 is often denoted by w (see, e.g., Ref. 7), while in the CP^2 case we have

$$\begin{aligned}
\partial f_2 &= A[(1+|f_2|^2)\varphi_2^2 + f_2\overline{f_3}\,\varphi_3^2],\\
\partial f_3 &= A[(1+|f_3|^2)\varphi_3^2 + f_3\overline{f_2}\,\varphi_2^2],\\
A &= 1+|f_2|^2+|f_3|^2.
\end{aligned} \tag{38}$$

Note that in Refs. 5 and 6 the functions f_2 and f_3 are denoted by w_1 and w_2, respectively.

Similarly,

$$\overline{\partial} f_i = A\,[\overline{\Phi}_i^2 + f_i\,\overline{f_k}\overline{\Phi}_k^2]. \tag{39}$$

III. THE WEIERSTRASS PROBLEM[1,3]

In the Weierstrass problem we consider two complex functions $\psi = \psi(\zeta,\overline{\zeta})$ and $\phi = \phi(\zeta,\overline{\zeta})$, which satisfy

$$\partial\,\psi = p\,\phi, \quad \overline{\partial}\phi = -p\,\psi, \quad p = |\phi|^2+|\psi|^2. \tag{40}$$

Note that we have not specified $\overline{\partial}\psi$, nor $\partial\phi$.

A natural question then arises. Is this problem related to the harmonic maps of the previous sections, presumably corresponding to the case of CP^1?

This is indeed the case as has been discussed in Ref. 4. To see this we put

$$w = \frac{\psi}{\overline{\phi}} \tag{41}$$

and note that

$$\psi = w\,\frac{(\overline{\partial}\overline{w})^{1/2}}{1+|w|^2}, \quad \phi = \frac{(\partial w)^{1/2}}{1+|w|^2} \tag{42}$$

satisfy (40). In fact, one can show that (19) and (40) are equivalent.

Moreover, we can introduce three real quantities:

$$\begin{aligned}
X_1 &= i\int_\gamma [\overline{\psi}^2+\phi^2]d\zeta - [\psi^2+\overline{\phi}^2]d\overline{\zeta},\\
X_2 &= \int_\gamma [\overline{\psi}^2-\phi^2]d\zeta + [\psi^2-\overline{\phi}^2]d\overline{\zeta},\\
X_3 &= -2\int_\gamma \overline{\psi}\phi d\zeta + \psi\overline{\phi}d\overline{\zeta},
\end{aligned} \tag{43}$$

where γ is any curve from a fixed point to ζ. Then, it is easy to show that if ψ and ϕ satisfy (40), the functions X_i do not depend on the contour of the curve γ but only on its endpoints.

Furthermore, if we treat X_i as components of a vector $\vec{r} = (X_1, X_2, X_3)$ and introduce the metric

$$g_{\zeta\zeta} = (\partial\vec{r},\partial\vec{r}), \quad g_{\overline{\zeta}\overline{\zeta}} = (\overline{\partial}\vec{r},\overline{\partial}\vec{r}), \quad g_{\zeta\overline{\zeta}} = (\partial\vec{r},\,\overline{\partial}\vec{r}), \tag{44}$$

we find that, for fields which solve (19) on S^2, only $g_{\zeta\bar{\zeta}}$ is nonzero and is given by

$$g_{\zeta\bar{\zeta}}=\frac{|\partial w|^2}{(1+|w|^2)^2}=|Dz|^2, \tag{45}$$

where $D=1/2(D_1-iD_2)$, with D_i defined in (5); i.e., is a covariant derivative evaluated with respect to ζ. Note that (45) is a term in the general expression for the energy density of the CP^1 map. However, as all harmonic maps on S^2 satisfy $w=w(\zeta)$,[7] we note that $g_{\zeta\bar{\zeta}}$ is the total energy density of this map. (We are assuming here that we are not dealing with antiholomorphic maps, as in this case we simply interchange the roles of ζ and $\bar{\zeta}$.)

IV. A GENERALIZED WEIERSTRASS REPRESENTATION IN R^M

Having observed that the Weierstrass problem is related to the equations of the CP^1 model, we have in Refs. 5 and 6 presented a CP^{N-1} generalization of the Weierstrass problem. Our generalization was based on the observation that for a generalized Weierstrass system in multi-dimensional spaces we need a set of φ_i and ψ_i which generalize the φ and ψ of the CP^1 case.

Then we noted that the quantities φ_i^2, $i=2,\ldots,N$, defined in (33), provide such a choice as (37) is a natural CP^{N-1} generalization of (42).

What should we take for the functions ψ_i? In Ref. 6 we argued that (41) suggested that we put

$$\psi_i=f_i\,\bar{\varphi}_i \tag{46}$$

with no summation over the indices $i=2,\ldots,N$. Then, to complete the generalization of the Weierstrass system in multi-dimensional spaces, we need relations which would be analogs of (40), i.e., we need to prescribe the first derivatives $\bar{\partial}\varphi_i$ and $\partial\psi_i$ in terms of φ_i and ψ_i.

Note that from (46) we have

$$\partial\psi_i=\partial(f_i\,\bar{\varphi}_i)=\partial f_i\,\bar{\varphi}_i+f_i\,\overline{(\bar{\partial}\varphi_i)}. \tag{47}$$

So, we are left to specify $\bar{\partial}\varphi_i$ in terms of φ_i, f_i and their derivatives. To do this, in Ref. 1, we noted that from (36) we had

$$\begin{aligned}\bar{\partial}\varphi_i^2&=2\,\frac{f_i(\bar{f}_l\,\partial f_l)}{A^3}(\bar{f}_k\,\bar{\partial} f_k+f_k\,\overline{\partial f_k})\\&\quad+\frac{1}{A^2}[(1+|f|^2)\partial\bar{\partial}f_i-(\bar{f}_k\bar{\partial}f_k)\partial f_i-(f_k\overline{\partial f_k})\partial f_i-\bar{\partial}f_i(\bar{f}_k\partial f_k)-f_i(\overline{\partial f_k}\partial f_k)-f_i(\bar{f}_k\,\partial\bar{\partial}f_k)].\end{aligned} \tag{48}$$

However, Eq. (10) allowed us to eliminate the second derivatives $\partial\bar{\partial}f_i$ from (48) and also we noted that all the terms involving first derivatives $\bar{\partial}f$ and $\overline{\partial f}$ in (48) canceled. Thus we ended up with a simple expression

$$\bar{\partial}\varphi_i=-\frac{\varphi_i}{2A}(f_k\,\overline{\partial f_k})-\frac{f_i}{2\varphi_iA^2}(\overline{\partial f_k}\partial f_k)+\frac{f_i}{2\varphi_iA^3}(\overline{\partial f_k}f_k)(\bar{f}_l\partial f_l). \tag{49}$$

Next, taking the complex conjugate of (37),

$$\overline{\partial f_k}=A\,[\bar{\varphi}_k^2+\bar{f}_k\,f_l\bar{\varphi}_l^2], \tag{50}$$

and using (50) we have found

$$\bar{\partial}\varphi_i=-\frac{1}{2}\left\{A\varphi_i(\bar{\varphi}\cdot\psi)+\frac{\psi_i}{\varphi_i\bar{\varphi}_i}[(\bar{\varphi}^2\cdot\varphi^2)+(\bar{\varphi}\cdot\psi)(\bar{\psi}\cdot\varphi)]\right\} \tag{51}$$

(no summation over i). The second pair of equations for ψ_i then followed from (47), i.e.,

$$\partial\psi_i=A\bar{\varphi}_i\varphi_i^2+\frac{1}{2}A\psi_i(\bar{\psi}\cdot\varphi)-\frac{1}{2}\frac{|\psi_i|^2}{|\varphi_i|^2\bar{\varphi}_i}[(\bar{\varphi}^2\cdot\varphi^2)+(\bar{\varphi}\cdot\psi)(\bar{\psi}\cdot\varphi)] \tag{52}$$

(no summation over i).

Thus, in Ref. 6 we proposed the following form of the generalized Weierstrass system: The **generalized Weierstrass system** in multi-dimensional space is a set of $(2N-2)$ complex functions φ_i and ψ_i, $i=2,\dots,N$, which obey the following system of equations (no summation over i):

$$\bar{\partial}\varphi_i=-\frac{1}{2}\left\{A\varphi_i(\bar{\varphi}\cdot\psi)+\frac{\psi_i}{\varphi_i\bar{\varphi}_i}[(\bar{\varphi}^2\cdot\varphi^2)+(\bar{\varphi}\cdot\psi)(\bar{\psi}\cdot\varphi)]\right\}$$

and

$$\partial\psi_i=A\bar{\varphi}_i\varphi_i^2+\frac{1}{2}A\psi_i(\bar{\psi}\cdot\varphi)-\frac{1}{2}\frac{|\psi_i|^2}{|\varphi_i|^2\bar{\varphi}_i}[(\bar{\varphi}^2\cdot\varphi^2)+(\bar{\varphi}\cdot\psi)(\bar{\psi}\cdot\varphi)], \tag{53}$$

where

$$A=1+\sum_{k=2}^{N}\frac{|\psi_k|^2}{|\varphi_k|^2}.$$

From our construction it is clear that the above system of equations is equivalent to the equations of the CP^{N-1} sigma model (10). Moreover, it is easy to check that the system of equations (53) reduces to Eq. (19) when $N=2$.

V. A MODIFIED GENERALIZED WEIERSTRASS REPRESENTATION

The generalized Weierstrass representation given in the previous section leads to pairs of functions φ_i, ψ_i, $i=2,\dots,N$, and, as discussed in Ref. 6, to a geometric interpretation in terms of surfaces in R^M for which their metric is given by $|Dz|^2$ [as in the CP^1 case—see (45)]. This is the case for the holomorphic solutions but we know[7] that CP^{N-1} models have harmonic maps which are not holomorphic (even CP^1 has such maps; in this case, antiholomorphic maps, but these can be considered to be complex conjugates of holomorphic ones). But for CP^{N-1}, $N>2$, we have also maps which are neither holomorphic nor antiholomorphic. So can we generalize the Weierstrass problem differently to bring out this property?

In fact, our discussion of the CP^{N-1} models does tell us what to do. We should use both φ_i and Φ_i. Thus we should consider a larger problem and use Φ_i, φ_i, and f_i.

Then taking (36) and repeating the steps as in (48) (and using φ^2 and Φ^2) we get

$$\bar{\partial}\varphi_i^2=-A\varphi_i^2(\varphi^{\dagger 2}\cdot f)-f_i[(\varphi^{\dagger 2}\cdot\varphi^2)+(f^\dagger\cdot\varphi^2)(\varphi^{2\dagger}\cdot f)], \tag{54}$$

$$\bar{\partial}\Phi_i^2=-A\Phi_i^2(f^\dagger\cdot\bar{\Phi}^2)-\bar{f}_i[(\Phi^{\dagger 2}\cdot\Phi^2)+(f^\dagger\cdot\bar{\Phi}^2)(\bar{\Phi}^{\dagger 2}\cdot f)]. \tag{55}$$

These equations should then be supplemented with the expressions for ∂f and $\partial\bar{f}$. The latter are given by (37) and (39) and so take the form

$$\partial f_i = A\,[\varphi_i^2 + f_i \overline{f_k} \varphi_k^2],$$
$$\overline{\partial} f_i = A\,[\overline{\Phi}_i^2 + f_i \overline{f_k} \overline{\Phi}_k^2], \tag{56}$$

where, as usual, $A = 1 + (f^\dagger \cdot f)$ and all indices, and summations, go over $(2,\ldots,N)$.

These four sets of equations (54)–(56), for three sets of complex functions, f_i, φ_j and Φ_k, constitute our **modified generalized Weierstrass problem**.

Let us make a few comments.

(i) The equations fall into two sets (those involving ∂f_i and φ_j and those involving $\partial \overline{f_i}$ and Φ_j). Both sets are equivalent to the same equations for f_i, namely, (10).

(ii) Instead of taking f_i we could introduce, in analogy with (46), new functions ψ_i and Ψ_i by, say, $\psi_i = f_i \overline{\varphi_i}$ and $\Psi_i = f_i \overline{\Phi}_i$. Then our set of functions would effectively decouple.

(iii) One can consider what happens when f_i are holomorphic; i.e., $\overline{\partial} f_i = 0$. Then, as is easy to check, $f^\dagger \cdot \overline{\Phi}^2 = 0$, which in turn shows that $|\Phi^2|^2 = 0$, thus demonstrating that $\Phi_i^2 = 0$, and we are left with (54) and (56) for f_i, φ_j, i.e., with the previous system (53).

VI. GEOMETRICAL ASPECTS

Next we consider some geometrical aspects of our procedure. This requires the introduction of a real vector $X_i \in R^M$ which is a generalization of the vector (43) constructed for CP^1. In Ref. 6 we have introduced such a vector and showed that its properties generalize those of (43).

However, our approach here generalizes the discussion in Ref. 6 and elucidates some of the points made there. Namely, in our new construction we exploit the matrices M and L. We introduce two matrices

$$V = \int_\gamma M\, d\overline{\zeta} + \int_\gamma M^\dagger\, d\zeta \tag{57}$$

and

$$W = \int_\gamma L\, d\overline{\zeta} + \int_\gamma L^\dagger\, d\zeta, \tag{58}$$

and for the components of our vectors we take individual entries of each matrix. As $\mathrm{Tr}\, M = \mathrm{Tr}\, L = 0$ we see that V and W have, each, only $N^2 - 1$ independent entries so our construction gives us two vectors in R^{N^2-1}.

Notice also that W and V do not depend on the contour of integration γ. This follows from the fact that for an integral

$$Z = \int_\Gamma F(\zeta,\overline{\zeta})\, d\zeta + \overline{F}(\zeta,\overline{\zeta})\, d\overline{\zeta}$$

to be independent of the integration contour Γ the condition is that F and $\overline{F}$ must satisfy

$$\overline{\partial} F = \partial \overline{F},$$

which is the case for V and W due to, respectively, our conservation laws (27) and (28).

Of course we can reexpress our vectors V and W in terms of the Weierstrass functions φ_i, Φ_j and f_k or in terms of φ_i, ψ_j, Φ_k, and Ψ_l.

Note that in the CP^1 case the matrix W is given by

$$W = -\frac{1}{(1+|w|^2)^2}\begin{pmatrix} w\overline{\partial w} & -\overline{\partial w} \\ w^2\overline{\partial w} & -w\overline{\partial w} \end{pmatrix}, \tag{59}$$

and so, given (42), we see that the integrands of the first terms in X_i of (43) have the form

$$x_1 = -i[\overline{L}_{21} - \overline{L}_{12}], \quad x_2 = -[\overline{L}_{21} + \overline{L}_{12}], \quad x_3 = -\overline{L}_{11} = \overline{L}_{22}. \tag{60}$$

So should we consider a new $2(N^2-1)$ vector, the first half of whose components are various entries of the matrix W, and the second half those of V? In the CP^1 case, as shown in Ref. 5, we can restrict ourselves to a vector with only three components. So we add both contributions and consider an N^2-1 component vector given by all the entries (except the top left hand corner one) of the matrix

$$X = \int_\gamma (M+L)\, d\overline{\zeta} + \int_\gamma (M^\dagger + L^\dagger)\, d\zeta. \tag{61}$$

Next we calculate the components of the induced metric

$$g_{\alpha\beta} = \sum_{lk} \frac{\partial X_{kl}}{\partial \alpha}\frac{\partial X_{lk}}{\partial \beta}, \tag{62}$$

where α and β are ζ or $\overline{\zeta}$. But

$$\frac{\partial X}{\partial \overline{\zeta}} = (M+L), \quad \frac{\partial X}{\partial \zeta} = (M^\dagger + L^\dagger), \tag{63}$$

where we are still using the matrix formulation. Hence

$$g_{\overline{\zeta\zeta}} = \mathrm{Tr}\,(M+L)^2, \quad g_{\zeta\zeta} = \mathrm{Tr}\,(M^\dagger + L^\dagger)^2, \quad g_{\zeta\overline{\zeta}} = \mathrm{Tr}\,[(M+L)(M^\dagger+L^\dagger)]. \tag{64}$$

However, given the form of M in (21) and L in (22), we see that

$$\mathrm{Tr}\,M^2 = \mathrm{Tr}\,L^2 = \mathrm{Tr}\,(M^\dagger)^2 = \mathrm{Tr}\,(L^\dagger) = \mathrm{Tr}\,L^\dagger M = \mathrm{Tr}\,M^\dagger L = 0, \tag{65}$$

and so we are left with

$$g_{\overline{\zeta\zeta}} = 2\,\mathrm{Tr}\,(ML), \quad g_{\zeta\zeta} = 2\,\mathrm{Tr}\,(M^\dagger L^\dagger), \quad g_{\zeta\overline{\zeta}} = \mathrm{Tr}\,[MM^\dagger + LL^\dagger]. \tag{66}$$

Next we observe that

$$\mathrm{Tr}\,MM^\dagger = \mathrm{Tr}\,(1-P)\frac{\overline{\partial} f \otimes f^\dagger}{|f|^2}\frac{f\otimes \partial f^\dagger}{|f|^2} = \frac{\partial f^\dagger \cdot \overline{\partial} f}{|f|^4} - \frac{(\partial f^\dagger \cdot f)(f^\dagger \cdot \overline{\partial} f)}{|f|^6} = |Dz|^2, \tag{67}$$

where, as in (45), D denotes the covariant derivative evaluated with respect to ζ. Similarly,

$$\mathrm{Tr}\,LL^\dagger = |\overline{D}z|^2, \tag{68}$$

where $\overline{D}$ is again the covariant derivative but this time evaluated with respect to $\overline{\zeta}$.

Note that, together, the two terms in $g_{\zeta\overline{\zeta}}$ give the total energy density of the map (i.e., $|Dz|^2 + |\overline{D}z|^2$).

What about $g_{\zeta\zeta}$ and $g_{\overline{\zeta\zeta}}$? They are given by

$$g_{\overline{\zeta\zeta}} = -\operatorname{Tr}(1-P)\frac{\bar{\partial}f\otimes f^{\dagger}\cdot f\otimes\bar{\partial}f^{\dagger}}{|f|^4} = -\operatorname{Tr}(1-P)\frac{\bar{\partial}f\otimes\bar{\partial}f^{\dagger}}{|f|^2}, \tag{69}$$

$$g_{\zeta\zeta} = -\operatorname{Tr}(1-P)\frac{\partial f\otimes f^{\dagger}\cdot f\otimes\partial f^{\dagger}}{|f|^4} = -\operatorname{Tr}(1-P)\frac{\partial f\otimes\partial f^{\dagger}}{|f|^2}, \tag{70}$$

and, at first sight, they appear to be nonvanishing. However, they do, in fact, vanish on the solutions of the CP^{N-1} model, i.e., on the vectors f which satisfy (10), at least those that are defined on S^2. To see this we note that

$$g_{\overline{\zeta\zeta}} = \frac{-|f|^2(\bar{\partial}f^{\dagger}\cdot\bar{\partial}f) + (\bar{\partial}f^{\dagger}\cdot f)(f^{\dagger}\cdot\bar{\partial}f)}{|f|^2} \tag{71}$$

and $g_{\zeta\zeta}$ is its complex conjugate.

However, all solutions of (10) defined on S^2 are[7] of the type

$$f = P_+^k g, \tag{72}$$

where g is a holomorphic vector, i.e., $g \neq g(\bar{\zeta})$, and k is some integer taken from the set $\{0,1,\ldots,N-1\}$, and $P_+^l g$ is defined by the successive, i.e., $P_+^l g = P_+(P_+^{l-1}g)$, repetition of the operation

$$P_+h = \partial h - h\frac{(h^{\dagger}\cdot\partial h)}{|h|^2}. \tag{73}$$

Then, as is known,[7] $P_+^k g$ satisfy

$$(P_+^l g)^{\dagger}\cdot P_+^k g = 0 \quad \text{if} \quad k \neq l, \tag{74}$$

$$\partial P_+^k g = P_+^{k+1}g + P_+^k g\frac{(P_+^k g^{\dagger}\cdot\partial P_+^k g)}{|P_+^k g|^2}, \tag{75}$$

$$\bar{\partial}P_+^k g = -P_+^{k-1}g\frac{|P_+^k g|^2}{|P_+^{k-1}g|^2}. \tag{76}$$

Thus $(\bar{\partial}f^{\dagger}\cdot\bar{\partial}f)=0$ and $(f^{\dagger}\cdot\bar{\partial}f)=0$ and so we see that $g_{\zeta\zeta}=0$ (and so also $g_{\overline{\zeta\zeta}}=0$).

VII. THE CP^1 CASE REVISITED

In the CP^1 case it is convenient to calculate its energy momentum tensor

$$T_{\mu\nu} = (D_\mu z)^{\dagger}\cdot D_\nu z + (D_\nu z)^{\dagger}\cdot D_\mu z - \delta_{\mu\nu}|D_\alpha|^2. \tag{77}$$

Then, as is known[7]

$$\bar{\partial}(T_{11}+iT_{12}) = 0 \tag{78}$$

and in the CP^1 case

$$J = T_{11}+iT_{12} = \frac{\partial w\,\bar{\partial}w}{[1+|w|^2]^2}. \tag{79}$$

When the CP^1 model is defined on S^2 we find that $J=0$, which shows that all the harmonic CP^1 maps on S^2 are either holomorphic or antiholomorphic,[7] but for the CP^1 model on R^2, or for CP^{N-1}, $N>2$, J does not have to vanish.

In the CP^1 model case we have (59),

$$W=-\frac{1}{(1+|w|^2)^2}\begin{pmatrix} w\bar{\partial}\overline{w} & -\bar{\partial}\overline{w} \\ w^2\bar{\partial}\overline{w} & -w\bar{\partial}\overline{w} \end{pmatrix} \tag{80}$$

and

$$V=\frac{\bar{\partial}w}{(1+|w|^2)^2}\begin{pmatrix} -\overline{w} & -\overline{w}^2 \\ 1 & \overline{w} \end{pmatrix}. \tag{81}$$

This allows us to express $\bar{\partial}w$ in terms of $\bar{\partial}\overline{w}$, J and p given by (40). We find

$$\bar{\partial}w=\frac{J(1+|w|^2)^2}{\partial w}=J\,\overline{\bar{\partial}w}\frac{(1+|w|^2)^2}{|\partial w|^2}. \tag{82}$$

However, using (42), we see that

$$p^2=\frac{|\partial w|^2}{(1+|w|^2)^2} \tag{83}$$

and so we have

$$\bar{\partial}w=\overline{\bar{\partial}w}\frac{J}{p^2}. \tag{84}$$

This allows us to combine the two vectors V and W into

$$V+W=\frac{\bar{\partial}\overline{w}}{(1+|w|^2)^2}\begin{pmatrix} -w-R\overline{w} & 1-R\overline{w}^2 \\ R-w^2 & w+R\overline{w} \end{pmatrix}, \tag{85}$$

where $R=J/p^2$.

This explains the origin of the expressions for the components of X_i given in Ref. 4. However, it is clear that this possibility to gather both terms into one expression does not generalize to higher CP^{N-1} models.

VIII. SUMMARY AND CONCLUDING REMARKS

The main aim of this article has been to derive a generalization of the Weierstrass system to the CP^{N-1} case for which the metric of the induced surfaces is determined by the energy density of the corresponding harmonic map.

This has led us to introduce a set of $3N$ complex functions φ_i, Φ_j and f_k which are required to satisfy a system of four classes of first order equations and which are equivalent to the full system of equations of the CP^{N-1} model.

We have also introduced a set of (N^2-1) real quantities X_i, which can be treated as coordinates of a surface immersed in R^{N^2-1} and we have shown that the induced metric of our map is given by

$$ds^2=(|Dz|^2+|\overline{D}z|^2)\,d\zeta\,d\overline{\zeta}. \tag{86}$$

The study of the generalized Weierstrass representations for surfaces immersed in multidimensional spaces was initiated by Konopelchenko and Landolfi.[3] Our work here, in which we have adopted an alternative approach based on the CP^{N-1} sigma models, provides a generalization of their results.

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